

About Factorial Sums

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Abstract. In this paper, we present some new inequalities for factorial sum.

Application 1. We have the following inequality

$$\sum_{k=1}^n k! \leq \frac{2((n+1)!-1)}{n+1}$$

Proof. If $x_k, y_k > 0$ ($k = 1, 2, \dots, n$), have the same monotonicity, then

$$\left(\frac{1}{n} \sum_{k=1}^n x_k \right) \left(\frac{1}{n} \sum_{k=1}^n y_k \right) \leq \frac{1}{n} \sum_{k=1}^n x_k y_k \quad (1)$$

the Chebishev's inequality.

If x_k, y_k have different monotonicity, then holds true the reverse inequality, we take $x_k = k, y_k = k!$ ($k = 1, 2, \dots, n$) and use that $\sum_{k=1}^n k \cdot k! = (n+1)! - 1$.

Application 2. We have the following inequality

$$\sum_{k=1}^n k! \leq \frac{3(n+1)(n+1)!}{n^2 + 3n + 5}$$

Proof. In (1) we take

$$x_k = k^2 + k + 1;$$

$$y_k = k! \quad (k = 1, 2, \dots, n)$$

and the identity

$$\sum_{k=1}^n (k^2 + k + 1)k! = (n+1)(n+1)!$$

Application 3. We have the following inequality

$$\sum_{k=1}^n \frac{1}{k!} \geq \frac{n^2(n+1)}{2((n+1)!-1)}$$

Proof. Using the Application 1, we take

$$\sum_{k=1}^n \frac{1}{k!} \geq \frac{n^2}{\sum_{k=1}^n k!} \geq \frac{n^2(n+1)}{2((n+1)!-1)}$$

Application 4. We have the following inequality

$$\sum_{k=1}^n \frac{1}{k!} \geq \frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

Proof. Using the Application 2, we take

$$\sum_{k=1}^n \frac{1}{k!} \geq \frac{n^2}{\sum_{k=1}^n k!} \geq \frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

Application 5. We have the following inequality:

$$\sum_{k=1}^n \frac{1}{k!} \geq 1 + \frac{2}{n} \left(1 - \frac{1}{n!} \right)$$

Proof. In (1) we take $x_k = k$, $y_k = \frac{1}{(k+1)!}$, $(k = 1, 2, \dots, n)$ and we obtain

$$\frac{1}{n} \left(\sum_{k=1}^n k \right) \left(\sum_{k=1}^n \frac{1}{(k+1)!} \right) \geq \sum_{k=1}^n \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$

therefore

$$\left(\sum_{k=1}^n \frac{1}{(k+1)!} \right) \geq \frac{2}{n+1} \left(1 - \frac{1}{(n+1)!} \right)$$

or

$$\sum_{k=2}^n \frac{1}{k!} \geq \frac{2}{n} \left(1 - \frac{1}{n!} \right)$$

therefore

$$\left(\sum_{k=1}^n \frac{1}{k!} \right) \geq 1 + \frac{2}{n} \left(1 - \frac{1}{n!} \right)$$

Application 6. We have the following inequality:

$$\left(\sum_{k=1}^n \frac{1}{(k+2)^2 k!} \right) \geq \frac{2}{n+5} \left(1 - \frac{1}{(n+2)!} \right)$$

Proof. In (1) we take $x_k = k+2$, $y_k = \frac{1}{(k+2)^2 k!}$, $(k = 1, 2, \dots, n)$

therefore

$$\frac{1}{n} \left(\sum_{k=1}^n (k+2) \right) \sum_{k=1}^n \frac{1}{(k+2)^2 k!} \geq \sum_{k=1}^n \frac{1}{(k+2)^2 k!} = 1 - \frac{1}{(n+2)!}$$

therefore

$$\sum_{k=1}^n \frac{1}{(k+2)^2 k!} \geq \frac{2}{n+5} \left(1 - \frac{1}{(n+2)!} \right)$$

Application 7. We have the following inequality:

$$\sum_{k=1}^n \frac{1}{k(k+1)(k+2)!} \geq \frac{6}{2n^2+9n+1} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)!} \right)$$

Proof. In (1) we take

$$x_k = k^2 + 2k + 2, \quad y_k = \frac{1}{k(k+1)(k+2)!}, \quad (k=1, 2, \dots, n)$$

then

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n (k^2 + 2k + 2) \sum_{k=1}^n \frac{1}{k(k+1)(k+2)!} &\geq \sum_{k=1}^n \frac{k^2 + 2k + 2}{k(k+1)(k+2)!} = \\ &= \sum_{k=1}^n \frac{1}{k(k+1)!} - \frac{1}{(k+1)(k+2)!} = \frac{1}{2} - \frac{1}{(n+1)(n+2)!} \end{aligned}$$

Application 8. We have the following inequality:

$$\sum_{k=1}^n \frac{1}{4k^4 + 1} \geq \frac{n}{2n^2 + 2n + 1}$$

Proof. In (1) we take $x_k = 4k$, $y_k = \frac{1}{4k^4 + 1}$, $(k=1, 2, \dots, n)$,

therefore

$$\frac{1}{n} \left(\sum_{k=1}^n 4k \right) \left(\sum_{k=1}^n \frac{1}{4k^4 + 1} \right) \geq \sum_{k=1}^n \frac{4k}{4k^4 + 1} = \sum_{k=1}^n \left(\frac{1}{2k^2 - 2k + 1} - \frac{1}{2k^2 + 2k + 1} \right) = \frac{2n(n+1)}{2n^2 + 2n + 1}$$

Application 9. We have the following inequality:

$$\sum_{k=1}^n \frac{1}{4k^4 - 1} \geq \frac{3n}{(2n+1)^2}$$

Proof. In (1) we take $x_k = k^2$, $y_k = \frac{1}{4k^2 - 1}$, $(k=1, 2, \dots, n)$ then

$$\frac{1}{n} \left(\sum_{k=1}^n k^2 \right) \left(\sum_{k=1}^n \frac{1}{4k^2 - 1} \right) \geq \sum_{k=1}^n \frac{k^2}{4k^2 - 1} = \frac{n(n+1)}{2(2n+1)}, \text{ etc.}$$

Reference:

[1] Octogon Mathematical Magazine (1993-2007)

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